Combinatorial Number Theory

Lecture Notes

Contents

Chapter 1

Ramsey's Theorem

1.1. Ramsey's Theorem for graphs

Definition 1. A *graph* $G = (V, E)$ is a set V of points, called *vertices*, and a set E of distinct pairs of vertices, called *edges*.

Definition 2. A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

Figure 1.1 below depicts a graph G with four vertices $V = \{V_1, V_2, V_3, V_4\}$ and four edges $E = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{V_1, V_2\}$, $e_2 = \{V_2, V_3\}$, $e_3 = \{V_3, V_4\}$, and $e_4 = \{V_2, V_4\}$. Note that edges are *unordered* pairs of vertices, meaning that $\{V_1, V_2\}$ and $\{V_2, V_1\}$ refer to the same edge. Next to it is a graph $G' = (V', E')$ with $V' = V = \{V_1, V_2, V_3, V_4\}$ and $E' = \{e_1, e_3\}$. Since $V' \subseteq V$ and $E' \subseteq E$, we deduce that G' is a subgraph of G.

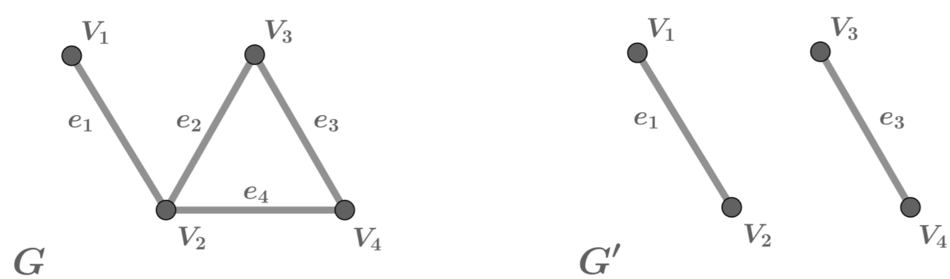

Figure 1.1: A graph G and one of its subgraphs G'.

Definition 3. Given $n \in \mathbb{N}$, a *complete graph on n vertices*, denoted by K_n, is a graph with n vertices and the property that every pair of distinct vertices is connected by an edge.

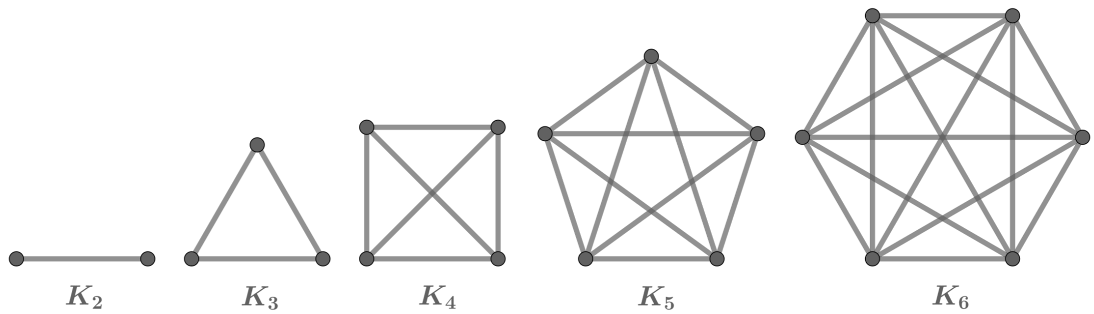

Figure 1.2: A depiction of K_n for $n = 2, 3, 4, 5$, and 6.

Definition 4. An *edge-coloring* of a graph $G = (V, E)$ is an assignment of a color to each edge of the graph. A graph that has been edge-colored is called *monochromatic* if all of its edges are the same color.

An edge-coloring of a graph can also be viewed as a function where the domain is the set of edges of the graph and the codomain is the set of colors. For example, suppose one has a graph with edges $E = \{e_1, e_2, e_3\}$ and a set of colors $C = \{\text{red}, \text{blue}\}$. A valid coloring of this graph can be seen as a function $\chi : E \to C$, where, for instance, $\chi(e_1) = \text{red}$, $\chi(e_2) = \text{blue}$, and $\chi(e_3) = \text{red}$.

Ramsey's Theorem for graphs. *For any $n, m \in \mathbb{N}$ there exists $R = R(n, m) \in \mathbb{N}$ such that any edge-coloring of K_R with at most m colors contains a monochromatic copy of K_n as a subgraph.*

Let us illustrate the content of Ramsey's Theorem for graphs by looking at an example. If the edge-coloring consists only of two colors, say red and blue, and we assume $n = 3$, then Ramsey's Theorem asserts that there exists a number $R(3, 2)$ such that any edge-coloring of a complete graph on $R(3, 2)$ vertices admits a monochromatic triangle. Note that $R(3, 2)$ cannot equal 5, because Figure 1.3 below shows a 2-coloring of K_5 containing no monochromatic triangle. However, taking

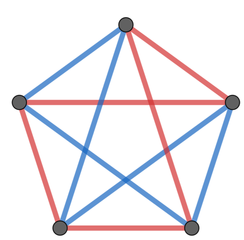

Figure 1.3: An edge-coloring of K_5 containing no monochromatic copy of K_3.

$R(3,2) = 6$ already works. Indeed, through some trial-and-error, one quickly realizes that it is impossible to find an edge-coloring of K_6 using only 2 colors that avoids monochromatic triangles. For instance, Figure 1.4 below shows a complete graph on 6 vertices where all but one edge have been colored either red or blue. As can be seen from the picture, it is impossible to complete the coloring without creating either a red or a blue triangle.

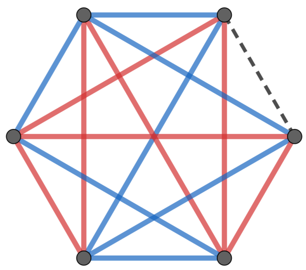

Figure 1.4: An almost-complete edge-coloring of K_6 that cannot be completed without creating a monochromatic copy of K_3. This example illustrates that it is impossible to color K_6 using two colors without producing a monochromatic copy of K_3.

The best possible value for $R(n,m)$ is called the *Ramsey number* for (n,m). Below is a list of Ramsey numbers known to date:

(n,m)	Ramsey Number
$(3,2)$	6
$(4,2)$	18
$(3,3)$	17
$(3,4)$	30
$(5,2)$	unknown
$(3,5)$	unknown
$(4,3)$	unknown
$\vdots$	

1.2. Ramsey's Theorem for 2-sets

Definition 5. A 2-*set* is a set consisting of exactly two elements. Given a set X, a 2-*subset* of X is any subset of X that is a 2-set. We will use $X^{(2)}$ to denote the set of all 2-subsets of X.

We have already seen examples of 2-subsets in the previous section. Indeed, the set of edges E of a graph $G = (V,E)$ consists of 2-subsets of the set of vertices V. In other words, $E \subseteq V^{(2)}$. Note that a graph $G = (V,E)$ is a complete graph if and only if $E = V^{(2)}$.

Definition 6. Let X be a set. A *coloring of* $X^{(2)}$ is an assignment of a color to each 2-subset of X. We call $X^{(2)}$ *monochromatic* if all elements in $X^{(2)}$ have the same color.

The following can be viewed as an "infinitary" version of Ramsey's Theorem for graphs.

Ramsey's Theorem for 2-sets. *Let X be an infinite set. Then for any finite coloring of $X^{(2)}$ there exists an infinite subset $Y \subseteq X$ such that $Y^{(2)}$ is monochromatic.*

Proof. Fix an arbitrary element $x_1 \in X$ and note that any 2-set of the form $\{x_1, x\}$ for $x \in X \setminus \{x_1\}$ has a certain color. Since the number of colors is finite but the set $X \setminus \{x_1\}$ is infinite, there exists an infinite subset $X_1 \subseteq X \setminus \{x_1\}$ such that all 2-sets of the form $\{x_1, x\}$ for $x \in X_1$ have the same color. Now fix an arbitrary element $x_2 \in X_1$ and let us repeat the same procedure. Any 2-set of the form $\{x_2, x\}$ for $x \in X_1 \setminus \{x_2\}$ has a certain color. For the same reason as before, since the number of colors is finite but the set $X_1 \setminus \{x_2\}$ is infinite, there exists an infinite subset $X_2 \subseteq X_1 \setminus \{x_1\}$ such all 2-sets of the form $\{x_2, x\}$ for $x \in X_2$ have the same color. Continuing this procedure produces an infinite sequence of distinct elements $x_1, x_2, x_3, \ldots$ and a nested family of infinite sets $X \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ such that for all $i \in \mathbb{N}$ we have $x_{i+1} \in X_i$ and the set $\{\{x_i, x\} : x \in X_i\}$ is monochromatic.

Let c_i denote the color of elements in the set $\{\{x_i, x\} : x \in X_i\}$. Then $c_1, c_2, c_3, \ldots$ is an infinite sequence of colors. Since there are only finitely many different colors, one color must appear infinitely often in this sequence. In other words, there exists a color c and an infinite sequence $i_1 < i_2 < i_3 < \ldots \in \mathbb{N}$ such that $c_{i_k} = c$ for all $k \in \mathbb{N}$.

To finish the proof, define $Y = \{x_{i_k} : k \in \mathbb{N}\}$ and observe that any 2-subset of Y is of the form $\{x_{i_k}, x_{i_\ell}\}$ for $k < \ell \in \mathbb{N}$. Since $x_{i_\ell} \in X_{i_\ell - 1}$ and $X_{i_\ell - 1} \subseteq X_{i_k}$, the 2-set $\{x_{i_k}, x_{i_\ell}\}$ has the color c. Hence all 2-subsets of Y have the color c, which proves that $Y^{(2)}$ is monochromatic. □

Proposition 7. *Ramsey's Theorem for 2-sets implies Ramsey's Theorem for graphs.*

Proof. We shall prove the contrapositive. Suppose $V_1, V_2, \ldots$ is an infinite sequence of distinct vertices and let K_R denote the complete graph on the vertices $V_1, \ldots, V_R$. If Ramsey's Theorem for graphs is false then for some $n, m \in \mathbb{N}$ and every $R \in \mathbb{N}$ there exists an edge-coloring $\chi_R : \{V_1, \ldots, V_R\}^{(2)} \to \{1, \ldots, m\}$ of K_R admitting no monochromatic copy of K_n.

If $s \leqslant R$ then any edge-coloring of K_R induces an edge-coloring of K_s, because K_s is a subgraph of K_R. In particular, we can restrict χ_R to K_s and obtain an edge-coloring of K_s with at most m colors admitting no monochromatic copy of K_n. Let us denote this restriction of χ_R to K_s by $\chi_{R,s}$.

Set $\mathscr{R}_1 = \mathbb{N}$. Consider the sequence of colors $(\chi_{R,2})_{R \in \mathscr{R}_1}$, all of which are edge-colorings of K_2. Since there are only finitely many possibilities of coloring the edges of K_2 with m colors and $\mathscr{R}_1$ is infinite, there exists an infinite subset $\mathscr{R}_2 \subseteq \mathscr{R}_1$ such that $(\chi_{R,2})_{R \in \mathscr{R}_2}$ all yield the same edge-coloring of K_2. Next, we can repeat the same argument with $\mathscr{R}_2$ in place of $\mathscr{R}_1$ and $\chi_{R,3}$ in place of $\chi_{R,2}$. Indeed, since there are only finitely many possibilities of coloring the edges of K_3 with m colors and $(\chi_{R,3})_{R\in\mathscr{R}_2}$ is an infinite sequence of edge-colorings of K_3, there exists an infinite subset $\mathscr{R}_3 \subseteq \mathscr{R}_2$ such that all colorings in $(\chi_{R,3})_{R\in\mathscr{R}_3}$ are identical. By continuing this procedure we end up with an infinite family of nested sets $\mathscr{R}_1 \supseteq \mathscr{R}_2 \supseteq \mathscr{R}_3 \supseteq \ldots$ such that all edge-colorings in $\{\chi_{R,s} : R \in \mathscr{R}_s\}$ are identical. In other words, for all $R_1, R_2 \in \mathscr{R}_s$ and all distinct $i, j \in \{1, \ldots, s\}$ the edge $\{V_i, V_j\}$ has the same color with respect to χ_{R_1} and χ_{R_2}.

Next define a finite coloring of $\mathbb{N}^{(2)}$ by assigning to each 2-subset $\{i, j\} \in \mathbb{N}^{(2)}$ the same color as the edge $\{V_i, V_j\}$ under the coloring χ_R, where R is any element in $\mathscr{R}_s$ and s is any number bigger than both i and j. Due to our construction, the choice of the color does not depend on which $R \in \mathscr{R}_s$ or which s bigger than i and j we choose. To finish the proof, note that with this coloring of $\mathbb{N}^{(2)}$ there does not exist a subset $Y \subseteq \mathbb{N}$ with $|Y| \geqslant n$ and such that $Y^{(2)}$ is monochormatic, because the existence of such a set would imply the existence of a monochromatic copy of K_n with respect to the coloring χ_R for sufficiently large R, which we know is not possible. This also means that there exists no infinite subset $Y \subseteq \mathbb{N}$ such that $Y^{(2)}$ is monochormatic, thus contradicting Ramsey's Theorem for 2-sets.

$\square$

1.3. Schur's Theorem

Fermat's Last Theorem states that for $m \geqslant 3$ the equation

$$x^m + y^m = z^m \tag{1.3.1}$$

has no positive integer solutions $x, y, z \in \mathbb{N}$. For centuries, this remained one of the biggest open problems in mathematics, and one whose intriguing nature captivated many mathematicians. Among them was also Issai Schur, who investigated a natural, localized version of Fermat's Last Theorem. More precisely, he wondered whether for any $m \geqslant 2$ the congruence equation

$$x^m + y^m \equiv z^m \pmod{p} \tag{1.3.2}$$

possesses non-trivial solutions for all but finitely many primes p. Note that any non-trivial solution to Fermat's equation $x^m + y^m = z^m$ also offers a non-trivial solution to Schur's equation $x^m + y^m \equiv z^m \pmod{p}$ for all primes p satisfying $p > z^m$, but not the other way around. In order to address (1.3.2), Schur proved a theorem that is often regarded as the earliest result in Ramsey Theory:

Schur's Theorem ([Sch17]). *For any $m \in \mathbb{N}$ there exists $S = S(m) \in \mathbb{N}$ such that if the set $\{1, 2, \ldots, S\}$ is colored using at most m colors then there exist monochromatic $x, y, z \in \{1, 2, \ldots, S\}$ with $x + y = z$.*

Proof. Take $S = R(3,m)$, where $R(3,m)$ is the Ramsey number for $(3,m)$. Let K_S denote the complete graph on S vertices and denote the vertices of K_S by $V_1, V_2, \dots, V_S$. Any coloring of the set $\{1,2,\dots,S\}$ induces an edge-coloring on K_S by assigning to each edge $\{V_i, V_j\}$ the color of the number $|i-j| \in \{1,2,\dots,S\}$. According to Ramsey's Theorem for graphs, K_S contains a monochromatic triangle. Let V_a, V_b, and V_c, for $a < b < c$, be the vertices of this monochromatic triangle. By setting

$$x = b - a, \qquad y = c - b, \qquad \text{and} \qquad z = c - a,$$

it is then easy to check that x, y, z have the same color and satisfy $x + y = z$. □

The smallest possible positive integer $S(m)$ for which the conclusion of Schur's Theorem holds is referred to as the *Schur number* for m. The known Schur numbers to date are:

m	Schur Number
2	5
3	14
4	45
5	161
6	unknown
7	unknown
$\vdots$	

Here is an example from Schur's original paper [Sch17] of a 3-coloring of $\{1,2,\dots,13\}$ admitting no monochromatic solution to the equation $x + y = z$:

color 1: $\{2,3,11,12\}$
color 2: $\{5,6,8,9\}$
color 3: $\{1,4,7,10,13\}$

More examples along these lines can be found here: https://oeis.org/A030126.

The proof that the Schur number for 5-colorings equals 161 took up 2 petabytes of space. Even though every 5-coloring of $\{1,\dots,161\}$ admits a monochromatic solution to $x + y = z$, there are 2447113088 many 5-colorings of $\{1,\dots,160\}$ admitting no monochromatic solution to $x + y = z$.

With the help of the above theorem, Schur was able to show that, contrary to Fermat's equation (1.3.1), its "local" counterpart (1.3.2) does possess non-trivial solutions.

Theorem 8. *Let $m \in \mathbb{N}$. There exists $F = F(m)$ such that for all prime numbers $p > F$ there exist $x, y, z \in \{1,2,\dots,p-1\}$ with $x^m + y^m \equiv z^m \pmod{p}$.*

For the proof of Theorem 8, we will need the following basic fact from algebra, the proof of which is left to the interested reader.

Lemma 9. *Let* $(K,+,\cdot)$ *be a field and* $f(x)\in K[x]$ *a polynomial of degree* $\deg(f)=m$ *with coefficients in* K*. Then the number of roots of* $f(x)$ *is at most* m*.*

Let us now see the proof of Theorem 8.

Proof of Theorem 8. Take $F=S(m)$, where $S(m)$ is as guaranteed by Schur's Theorem. Let p be any prime number bigger than F. The set $\mathbb{F}_p=\{0,1,\dots,p-1\}$ of congruence classes modulo p naturally forms a field $(\mathbb{F}_p,+,\cdot)$ under the modular arithmetic operations $+$ and $\cdot$. Let $\mathbb{F}_p^\times=\mathbb{F}_p\backslash\{0\}$ and consider the set

$$C:=\{x^m : x\in\mathbb{F}_p^\times\}.$$

Note that C is a subgroup of the multiplicative group $(\mathbb{F}_p^\times,\cdot)$. This means that $\mathbb{F}_p^\times$ can be covered by cosets of C. More precisely, there exist coset representatives $g_1,g_2,\dots,g_r\in\mathbb{F}_p^\times$ such that

$$\mathbb{F}_p^\times=g_1C\cup g_2C\cup\ldots\cup g_rC. \tag{1.3.3}$$

It follows from Lemma 9 that for any $y\in\mathbb{F}_p^\times$ the equation $x^m\equiv y \pmod p$ has at most m solutions, because the polynomial x^m-y can have no more than m roots. So any $y\in\mathbb{F}_p^\times$ admits at most m representation of the form x^m, which implies that that $m|C|\geqslant|\mathbb{F}_p^\times|$. It follows that C can have at most m cosets, or in other words, $r\leqslant m$. Since $p>F$, the set $\{1,\dots,F\}$ is a subset of $\mathbb{F}_p^\times=\{1,2,\dots,p-1\}$ and hence (1.3.3) yields a partition of the set $\{1,\dots,F\}$ involving r disjoint cells. We can think of this partition as a coloring of $\{1,\dots,F\}$ using r colors. Since $F=S(m)$ and $r\leqslant m$, it follows from Schur's Theorem that there exist monochromatic $\tilde x,\tilde y,\tilde z\in\{1,2,\dots,F\}$ for which $\tilde x+\tilde y=\tilde z$. Since $\tilde x,\tilde y,\tilde z$ have the same color, they all belong to the same coset. In other words, there exists a coset representative $g_i\in\{g_1,\dots,g_r\}$ such that $\tilde x,\tilde y,\tilde z\in g_iC$. Take any $x,y,z\in\mathbb{F}_p^\times$ for which

$$\tilde x\equiv g_ix^m \pmod p,\qquad \tilde y\equiv g_iy^m \pmod p,\qquad\text{and}\qquad \tilde z\equiv g_iz^m \pmod p,$$

which is possible because $\tilde x,\tilde y,\tilde z\in g_iC$. Then we have

$$g_ix^m+g_iy^m\equiv g_iz^m \pmod p,$$

from which it follows that

$$x^m+y^m\equiv z^m \pmod p,$$

because $g_i\not\equiv 0 \pmod p$. □

1.4. Ramsey's Theorem for k-sets

Definition 10. A *k-set* is a set consisting of exactly k elements. Given a set X, a *k-subset* of X is any subset of X that is a k-set. We will use $X^{(k)}$ to denote the set of all k-subsets of X.

We have already seen Ramsey's Theorem for 2-sets. Here is Ramsey's result in full generality.

Ramsey's Theorem for k-sets ([Ram30]). *Let X be an infinite set and $k \geqslant 2$. Then for any finite coloring of $X^{(k)}$ there exists an infinite subset $Y \subseteq X$ such that $Y^{(k)}$ is monochromatic.*

Proof. Let us use a proof by induction on k. The base case of the induction, when $k = 2$, follows from Ramsey's Theorem for 2-sets established in Section 1.2. To prove the inductive step, assume $k \geqslant 3$ and Ramsey's Theorem has already been proven for $(k-1)$-sets. Let $Y_0 = X$ and fix an arbitrary element $y_1 \in Y_0$. Note that any k-set of the form $\{y_1, x_2, \ldots, x_k\}$ for $\{x_2, \ldots, x_k\} \in (Y_0 \backslash \{y_1\})^{(k-1)}$ has a certain color, which induces a finite coloring on $(Y_0 \backslash \{y_1\})^{(k-1)}$. Applying Ramsey's Theorem for $(k-1)$-sets, we can find an infinite subset $Y_1 \subseteq Y_0 \backslash \{y_1\}$ such that all k-sets of the form $\{y_1, x_2, \ldots, x_k\}$ for $\{x_2, \ldots, x_k\} \in Y_1^{(k-1)}$ are monochromatic. Next, fix an arbitrary element $y_2 \in Y_1$ and repeat the same procedure. The given coloring of k-sets of the form $\{y_2, x_2, \ldots, x_k\}$ for $\{x_2, \ldots, x_k\} \in (Y_1 \backslash \{y_2\})^{(k-1)}$ induces a finite coloring of $(Y_1 \backslash \{y_2\})^{(k-1)}$. Applying Ramsey's Theorem for $(k-1)$-sets once more yields an infinite subset $Y_2 \subseteq Y_1 \backslash \{y_2\}$ such that all k-sets of the form $\{y_2, x_2, \ldots, x_k\}$ for $\{x_2, \ldots, x_k\} \in Y_2^{(k-1)}$ are monochromatic. Continuing this procedure produces an infinite sequence of distinct elements $y_1, y_2, y_3, \ldots$ and a nested family of infinite sets $X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$ such that for all $i \in \mathbb{N}$ the set $\{\{y_i, x_2, \ldots, x_k\} : \{x_2, \ldots, x_k\} \in Y_i^{(k-1)}\}$ is monochromatic. Moreover, we have $y_{i+1} \in Y_i$ for all $i \in \mathbb{N}$.

Let c_i denote the color of elements in the set $\{\{y_i, x_2, \ldots, x_k\} : \{x_2, \ldots, x_k\} \in Y_i^{(k-1)}\}$. Since the sequence $c_1, c_2, c_3, \ldots$ is infinite but the number of colors is finite, one color must appear infinitely often in this sequence. In other words, there exists a color c and an infinite subsequence $c_{i_1}, c_{i_2}, c_{i_3}, \ldots \in \mathbb{N}$ such that $c_{i_\ell} = c$ for all $\ell \in \mathbb{N}$. To finish the proof, define $Y = \{y_{i_k} : k \in \mathbb{N}\}$ and observe that any k-subset of Y is of the form $\{y_{i_{\ell_1}}, \ldots, y_{i_{\ell_k}}\}$ for $\ell_1 < \ldots < \ell_k \in \mathbb{N}$. Since $\{y_{i_{\ell_2}}, \ldots, y_{i_{\ell_k}}\} \in Y_{i_{\ell_1}}$ because $\ell_1 < \ell_2 < \ldots < \ell_k$, the k-set $\{y_{i_{\ell_1}}, \ldots, y_{i_{\ell_k}}\}$ has the color c. Hence all k-subsets of Y have the color c, which proves that $Y^{(k)}$ is monochromatic. $\square$

1.5. The compactness principle

Compactness Theorem for finite colorings. *Let Y be an infinite set, let $m \in \mathbb{N}$, and let $\mathscr{F}$ be a collection of finite subsets of Y. The following are equivalent:*

(i) For any coloring of Y using no more than m colors there exists $A \in \mathscr{F}$ such that all elements in A have the same color.

(ii) There exists a finite set $Z \subseteq Y$ such that for any finite coloring of Z using no more than m colors there exists $A \in \mathscr{F}$ with $A \subseteq Z$ and such that all elements in A have the same color.

Proof. The implication (ii) $\Longrightarrow$ (i) is immediate, so it only remains to prove (i) $\Longrightarrow$ (ii). We can view a coloring of Y that uses no more than m colors as a function $\chi\colon Y \to \{1,\ldots,m\}$ simply by associating a number from 1 to m with each color. This means the space of all possible colorings of Y can be identified with the product space $\{1,\ldots,m\}^Y$. Note that the finite set $\{1,\ldots,m\}$, endowed with the discrete topology, is a compact Hausdorff space. By Tychonoff's theorem, $\{1,\ldots,m\}^Y$ endowed with the product topology is therefore also a compact Hausdorff space.

For any finite non-empty set $Z \subseteq Y$ let $\mathscr{C}_Z$ be the set of all colorings in $\{1,\ldots,m\}^Y$ for which there is monochromatic $A \in \mathscr{F}$ with $A \subseteq Z$. Then $\mathscr{C}_Z$ is an open set in the product topology on $\{1,\ldots,m\}^Y$. Moreover, in light of statement(i), we have

$$\bigcup_{\substack{Z\subseteq Y\\ 0<|Z|<\infty}} \mathscr{C}_Z = \{1,\ldots,m\}^Y.$$

By compactness, it follows that there is some finite non-empty set $Z \subseteq Y$ such that $\mathscr{C}_Z = \{1,\ldots,m\}^Y$, completing the proof. □

1.6. Ramsey's Theorem for hypergraphs

A hypergraph is a generalization of a graph in which an edge can join multiple vertices at once.

Definition 11. Let $k \in \mathbb{N}$. A *k-uniform hypergraph* is a pair $G = (V,E)$ where V is a set of points, called *vertices*, and $E \subseteq V^{(k)}$ is a set of k-subsets of V, called *hyperedges*.

Given $k,n \in \mathbb{N}$ with $k \leqslant n$, a *complete k-uniform hypergraph on n vertices* is a k-uniform hypergraph $G = (V,E)$ where the set of vertices has cardinality n and where every set of k distinct vertices in V is connected by an edge. In other words, $G = (V,E)$ is a complete k-uniform hypergraph on n vertices if $|V| = n$ and $E = V^{(k)}$.

Ramsey's Theorem for hypergraphs. *For any $n,m,k \in \mathbb{N}$ there exists a number $R = R_k(n,m) \in \mathbb{N}$ such that any edge-coloring of a complete k-uniform hypergraph on R vertices with at most m colors admits a monochromatic copy of a complete k-uniform hypergraph on n vertices.*

Proof. Let $n,m,k \in \mathbb{N}$ be given. If follows from Ramsey's Theorem for k-sets that for any m-coloring of $\mathbb{N}^{(k)}$ there exists a set $S \subseteq \mathbb{N}$ with $|S| = n$ such that $S^{(k)}$ is monochromatic. If we now apply the Compactness Theorem for finite colorings to this statement (with $Y = \mathbb{N}^{(k)}$ and $\mathscr{F} = \{S^{(k)} : S \subseteq \mathbb{N},\ |S| = n\}$), it follows that there exists some integer $R = R_k(n,m)$ such that for any m-coloring of $\{1,\ldots,R\}^{(k)}$ exists a set $S \subseteq \{1,\ldots,R\}$ with $|S| = n$ such that $S^{(k)}$ is monochromatic. But note that $\{1,\ldots,R\}^{(k)}$ can be identified with a complete k-uniform hypergraph on R vertices, and $S^{(k)}$ with a complete k-uniform hypergraph on n vertices. This finishes the proof. □

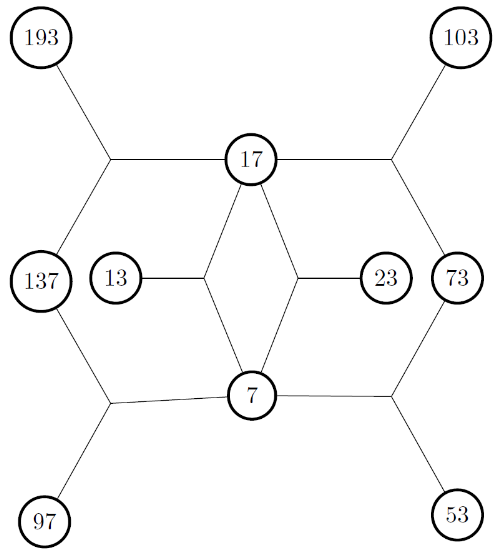

Figure 1.5: Here is an example of a 3-uniform hypergraph with vertices $V = \{7,13,17,23,53,73,97,103,137,193\}$, where three vertices are connected by a hyperedge if and only if their squares form a 3-term arithmetic progression. For example, $\{7,13,17\}$ is an edge, because $7^2, 13^2, 17^2$ are in an arithmetic progression.

1.7. Erdős-Szekeres' Theorem on convex polygons

Definition 12. A non-empty set $C \subseteq \mathbb{R}^2$ is called *convex* if for any $\vec{x}, \vec{y} \in C$ and $\lambda \in [0,1]$ one has $\lambda\vec{x} + (1-\lambda)\vec{y} \in C$.

The point $\lambda\vec{x} + (1-\lambda)\vec{y}$ is usually referred to as a *convex combination* of the points $\vec{x}$ and $\vec{y}$. Also observe that the set $\{\lambda\vec{x} + (1-\lambda)\vec{y} : \lambda \in [0,1]\}$ is just an algebraic description for the line segment joining the points $\vec{x}$ and $\vec{y}$.

Figure 1.6: A convex polygon (left) and a non-convex polygon (right).

Definition 13. The *convex hull* of a non-empty set $K \subseteq \mathbb{R}^2$ is the smallest convex set that contains K.

Since the intersection of convex sets is again a convex set, it follows that the convex hull of K equals the intersection of all convex sets that contain K. The convex hull can also be described algebraically as the set of all finite convex combinations of elements in the set. More precisely, if K is a subset of $\mathbb{R}^2$ and we use $\operatorname{conv}(K)$ to denote its convex hull, then

$$\operatorname{conv}(K) = \{w_1\vec{z}_1 + \ldots + w_\ell\vec{z}_\ell : \ell \in \mathbb{N},\ \vec{z}_1,\ldots,\vec{z}_\ell \in K,\ w_1,\ldots,w_\ell \in [0,1],\ w_1 + \ldots + w_\ell = 1\}. \tag{1.7.1}$$

Mind that the convex hull of K should not be confused with the *closed convex hull* of K, which is defined as the smallest closed convex set that contains K, and is usually denoted by $\overline{\operatorname{conv}}(K)$ instead of $\operatorname{conv}(K)$.

Definition 14. A non-empty set of points $K \subseteq \mathbb{R}^2$ is said to be in *convex position* if no point $\vec{x} \in K$ belongs to the convex hull of $K \setminus \{\vec{x}\}$.

For example, a finite set $K \subseteq \mathbb{R}^2$ is in convex position if and only if its elements are the corners of a convex polygon.

Definition 15. A set $K \subseteq \mathbb{R}^2$ is called *discrete* if it has no accumulation points.

Erdős-Szekeres' Theorem on points in convex position. *Let K be an infinite discrete set of points in $\mathbb{R}^2$. Then either there is an infinite subset of K whose points lie on a straight line or there is an infinite subset of K whose points are in convex position.*

For the proof of Erdős-Szekeres' Theorem on points in convex position we will need the following classical result from convex geometry.

Carathéodory's theorem. *Let $K \subseteq \mathbb{R}^2$ with $|K| \geqslant 4$ be given. Then K is in convex position if and only if any four distinct points from K form a convex quadrilateral.*

Proof. Clearly, if K is in convex position then any quadrilateral formed using points from K is convex. To prove the converse, we will show that if K is not in convex position then there exist four points in K such that one of these points lies within the triangle spanned by the others.

Suppose K is not in convex position. Then there exists a point $\vec{x} \in K$ lying in the convex hull of $K' = K \setminus \{\vec{x}\}$. In light of (1.7.1), this means that we can write $\vec{x}$ as

$$\vec{x} = w_1\vec{z}_1 + \ldots + w_\ell\vec{z}_\ell, \tag{1.7.2}$$

where $\vec{z}_1,\ldots,\vec{z}_\ell \in K'$ and $w_1,\ldots,w_\ell \in [0,1]$ with $w_1 + \ldots + w_\ell = 1$. Note that we can assume without loss of generality that $\vec{z}_1,\ldots,\vec{z}_\ell$ are in convex position. Indeed, if for example $\vec{z}_\ell$ belongs to the convex hull of $\vec{z}_1,\ldots,\vec{z}_{\ell-1}$ then we can express $\vec{z}_\ell$ as a convex combination of $\vec{z}_1,\ldots,\vec{z}_{\ell-1}$ and substitute this representation in (1.7.2), allowing us to represent $\vec{x}$ as a convex combination of $\vec{z}_1,\ldots,\vec{z}_{\ell-1}$ instead of $\vec{z}_1,\ldots,\vec{z}_\ell$. Thus, invoking induction on ℓ, we may assume that $\vec{z}_1,\ldots,\vec{z}_\ell$ are in convex position. This implies that $\vec{z}_1,\ldots,\vec{z}_\ell$ form the corners of a convex polygon. Since $\vec{x}$ lies inside this polygon and since convex polygons decompose into triangles (as illustrated in

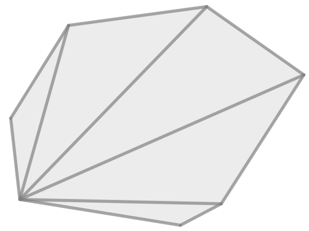

Figure 1.7: A convex polygon divided into triangles.

Figure 1.7), there exists $i < j < k \in \{1,\dots,\ell\}$ such that $\vec{x}$ lies in the triangle spanned by $\vec{z}_i, \vec{z}_j, \vec{z}_k$, finishing the proof. □

Proof of Erdős-Szekeres' Theorem on points in convex position. Let $K \subseteq \mathbb{R}^2$ be infinite. We begin by coloring $K^{(3)}$ by assigning the color red to $\{\vec{x}, \vec{y}, \vec{z}\} \in K^{(3)}$ if the points $\vec{x}, \vec{y}, \vec{z}$ are collinear and the color blue otherwise. According to Ramsey's Theorem for k-sets, there exists an infinite set $L \subseteq K$ such that all 3-sets in $L^{(3)}$ have the same color. If this color is red, then any three distinct points in L are collinear. This can only happen if all the points in L lie on a straight line, in which case we are done.

It remains to deal with the case when all elements in $L^{(3)}$ are blue, i.e., when no three points in L are collinear. In this situation, we need to apply Ramsey's Theorem one more time. Note that L is a discrete set. This implies that for any three points $\vec{x}, \vec{y}, \vec{z} \in L$ the triangle $\triangle\vec{x}\vec{y}\vec{z}$ contains only finitely many points from L. Color all elements in $L^{(3)}$ by assigning the color red to the 3-set $\{\vec{x}, \vec{y}, \vec{z}\} \in L^{(3)}$ if the triangle $\triangle\vec{x}\vec{y}\vec{z}$ contains an even number of points from L, and the color blue otherwise. By Ramsey's Theorem for k-sets there exists an infinite set $C \subseteq L$ such that $C^{(3)}$ is monochromatic. We claim that C is in convex position. Indeed, if C were not in convex position then, in view of Carathéodory's theorem, there exist four points $\vec{w}, \vec{x}, \vec{y}, \vec{z} \in C$ such that $\vec{w}$ lies inside the triangle $\Delta_0 = \triangle\vec{x}\vec{y}\vec{z}$. Note that Δ_0 splits into three smaller triangles, $\Delta_1 = \triangle\vec{w}\vec{y}\vec{z}$, $\Delta_2 = \triangle\vec{w}\vec{x}\vec{z}$, and $\Delta_3 = \triangle\vec{w}\vec{x}\vec{y}$, as seen in Figure 1.8. For $i = 0,1,2,3$ let $\#\Delta_i$ denote the number of points from L inside the

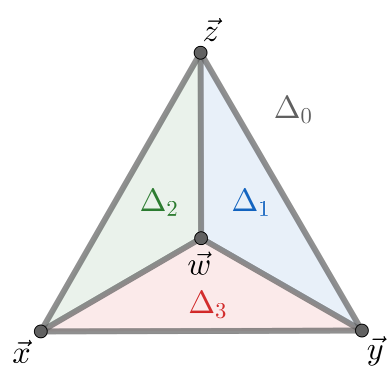

Figure 1.8

triangle Δ_i. Since no three points from L are collinear, there are no points on the boundary of any of these triangles aside from their corners. This means that the number of points from L inside Δ_0 equals the combined number of points inside the three smaller triangles plus the point $\vec{w}$, or in other words,

$$\#\Delta_0 = \#\Delta_1 + \#\Delta_2 + \#\Delta_3 + 1. \tag{1.7.3}$$

Recall that $C^{(3)}$ is monochromatic. If all elements in $C^{(3)}$ are red then the quantities $\#\Delta_0$, $\#\Delta_1$, $\#\Delta_2$, and $\#\Delta_3$ are even numbers. This would imply that the left hand side of (1.7.3) is an even number whereas the right hand side is an odd number, a contradiction. Similarly, if all elements in $C^{(3)}$ are blue then $\#\Delta_0$, $\#\Delta_1$, $\#\Delta_2$, $\#\Delta_3$ are odd numbers, implying that the left hand side of (1.7.3) is odd whereas the right hand side is even. Either way, we have obtained a contradiction, which means that C is in convex position. □

The following is a big open conjecture at the interface of convex geometry and Ramsey theory, posed by Erdős and Szekeres in 1960.

Conjecture (Erdős-Szekeres convex polygon problem)**.** *Let $n \geqslant 3$. Any set of $2^{n-2}+1$ points in the plane, no three of which are collinear, contains a subset of n points in convex position.*

1.8. Erdős-Szekeres' Theorem on monotone paths

Erdős-Szekeres' Theorem on monotone paths. *Fix $n, m \in \mathbb{N}$. Any sequence of distinct real numbers of length at least $nm+1$ admits either a monotonically increasing subsequence of length $n+1$ or a monotonically decreasing subsequence of length $m+1$.*

Proof. Let $x_1, x_2, \ldots, x_{nm+1}$ be a sequence of real numbers of length $nm+1$. Label each element x_i in the sequence with the pair (a_i, b_i), where a_i is the length of the longest monotonically increasing subsequence ending with x_i and b_i is the length of the longest monotonically decreasing subsequence ending with x_i. Note that any two elements in the sequence are labeled with a different pair: if $i < j$ and $x_i < x_j$ then $a_i < a_j$, and on the other hand if $x_i > x_j$ then $b_i < b_j$. If $a_i \leqslant n$ and $b_i \leqslant m$ for all i then there are only nm possible labels, contradicting the fact that there are $nm+1$ elements in the sequence each with a unique label. It follows that either $a_i > n$ or $b_i > m$ for some i, yielding either an increasing sequence of length at least $n+1$ or a decreasing sequence of length at least $m+1$. □

Chapter 2

Van der Waerden's Theorem

2.1. Notions of largeness

The goal of this section is to develop a general framework for dealing with notions of largeness for sets. In what follows, let X be a set and $\mathscr{P}$ a family of subsets of X. Since any reasonable notion of largeness is closed under supersets, the following definition will be very useful for our purposes.

Definition 16. We call $\mathscr{P}$ *upward closed* if for all $A \subseteq B \subseteq X$ we have $A \in \mathscr{P} \implies B \in \mathscr{P}$.

Natural examples of upward closed families include the set of all infinite subsets and the set of all cofinite subsets of a given infinite set X,

$$\mathscr{P}_{\text{inf}} = \{A \subseteq X : A \text{ is infinite}\} \qquad \text{and} \qquad \mathscr{P}_{\text{cofin}} = \{A \subseteq X : A \text{ is cofinite}\}.$$

Another example of an upward closed family is the collection of all sets that share a common point,

$$\mathscr{P}_{\{x\}} = \{A \subseteq X : x \in A\}$$

where $x \in X$ is fixed.

Definition 17. The *dual family* of $\mathscr{P}$, denoted by $\mathscr{P}^*$, is defined as

$$\mathscr{P}^* = \{A \subseteq X : A \cap B \neq \emptyset \text{ for all } B \in \mathscr{P}\}.$$

The families $\mathscr{P}_{\text{inf}}$ and $\mathscr{P}_{\text{cofin}}$ are mutually dual, meaning that $\mathscr{P}^*_{\text{inf}} = \mathscr{P}_{\text{cofin}}$ and $\mathscr{P}^*_{\text{cofin}} = \mathscr{P}_{\text{inf}}$, whereas the family $\mathscr{P}_{\{x\}}$ is self-dual in the sense that $\mathscr{P}_{\{x\}} = \mathscr{P}^*_{\{x\}}$. Note that if $\mathscr{P}$ is upward closed then its dual $\mathscr{P}^*$ is also upward closed. Also, if $\mathscr{P}$ is upward closed then we have the following two convenient properties:

- For any set $A \subseteq X$,

$$A \in \mathscr{P}^* \iff A^c \notin \mathscr{P}, \tag{2.1.1}$$

where $A^c = X \setminus A$ denotes the complement of A in X.
- $\mathscr{P}^{**} = \mathscr{P}$.

Definition 18. The family $\mathscr{P}$ is called *partition regular* if for any finite coloring of a set $A \in \mathscr{P}$ there exists a monochromatic subset of A that belongs to $\mathscr{P}$.

Using a standard "color blindness" argument, we deduce that any upward closed family $\mathscr{P}$ is partition regular if and only if for any disjoint $A,B \subseteq X$ with $A \cup B \in \mathscr{P}$ either $A \in \mathscr{P}$ or $B \in \mathscr{P}$. With some additional work, one can even remove the word disjoint from this statement.

Definition 19. We say a family of sets $\mathscr{P}$ is *closed under finite intersections* if for any $A_1,\dots,A_k \in \mathscr{P}$ we have $A_1 \cap \dots \cap A_k \in \mathscr{P}$.

Coming back to our previous examples, we see that the family $\mathscr{P}_{\text{inf}}$ is partition regular but not closed under finite intersections, whereas the family $\mathscr{P}_{\text{cofin}}$ is not partition regular but closed under finite intersections. In contrast, the family $\mathscr{P}_{\{x\}}$ is simultaneously partition regular and closed under finite intersections. These observations are explained by the next proposition.

Proposition 20. *Let $\mathscr{P}$ be an upward closed family of subsets of a set X. Then $\mathscr{P}$ is partition regular if and only if $\mathscr{P}^*$ is closed under finite intersections.*

Proof. ($\Rightarrow$) Suppose $\mathscr{P}$ is partition regular, let $A_1,\dots,A_k \in \mathscr{P}^*$, and define $C_i = A_i^c$ for $i = 1,\dots,k$. In view of (2.1.1) we have $C_1,\dots,C_k \notin \mathscr{P}$. As $\mathscr{P}$ is partition regular, it follows from $C_1,\dots,C_k \notin \mathscr{P}$ that $\bigcup_{i=1}^k C_i \notin \mathscr{P}$. Using (2.1.1) once more we get

$$\left(\bigcup_{i=1}^k C_i\right)^c = \bigcap_{i=1}^k A_i \notin \mathscr{P}^*.$$

This proves that $\mathscr{P}^*$ is closed under finite intersections.

($\Leftarrow$) Assume $\mathscr{P}^*$ is closed under finite intersections, let $C_1,\dots,C_k \in \mathscr{P}$, and assume $\bigcup_{i=1}^k C_i \in \mathscr{P}$. Define $A_i = C_i^c$ for $i = 1,\dots,k$ and note that from (2.1.1) and $\bigcup_{i=1}^k C_i \in \mathscr{P}$ we have

$$\bigcap_{i=1}^k A_i \notin \mathscr{P}^*.$$

Since $\mathscr{P}^*$ is closed under finite intersections, it follows that for some $i \in \{1,\dots,k\}$ we must have $A_i \notin \mathscr{P}^*$. By (2.1.1) we conclude that $C_i \in \mathscr{P}$, showing that $\mathscr{P}$ is partition regular. □

Proposition 21. *Let $\mathscr{P}$ be upward closed. Then the family $\mathscr{P} \wedge \mathscr{P}^* = \{A \cap B : A \in \mathscr{P},\ B \in \mathscr{P}^*\}$ is partition regular.*

Proof. Suppose $C \in \mathscr{P} \wedge \mathscr{P}^*$. It suffices to show that if $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$ then either $C_1 \in \mathscr{P} \wedge \mathscr{P}^*$ or $C_2 \in \mathscr{P} \wedge \mathscr{P}^*$. Pick $A \in \mathscr{P}$ and $B \in \mathscr{P}^*$ such that $C = A \cap B$, and define $D = C_1 \cup A^c$. If $D \in \mathscr{P}^*$ then $C_1 = A \cap D$ belongs to $\mathscr{P} \wedge \mathscr{P}^*$ and we are done. On the other hand, if $D \notin \mathscr{P}^*$ then $D^c \in \mathscr{P}$ (by (2.1.1)) and $C_2 = D^c \cap B$, which implies $C_2 \in \mathscr{P} \wedge \mathscr{P}^*$ and we are also done. □

2.2. Syndetic sets and thick sets

In what follows, let $A - n = \{m \in \mathbb{N} : m + n \in A\}$.

Definition 22. A set of positive integers $S \subseteq \mathbb{N}$ is called *syndetic* if there exists $h \in \mathbb{N}$ such that $S \cup (S-1) \cup \ldots \cup (S-h) = \mathbb{N}$.

Observe that syndetic sets are characterised by the property that the distance between consecutive elements is bounded. In other words, if $s_1 < s_2 < \ldots$ is an increasing enumeration of elements in S then S is syndetic if and only if $\sup_{k\in\mathbb{N}}(s_{k+1} - s_k) < \infty$.

Definition 23. A set of positive integers $T \subseteq \mathbb{N}$ is called *thick* if for every $h \in \mathbb{N}$ the intersection $T \cap (T-1) \cap \ldots \cap (T-h)$ is non-empty.

Thick sets are characterized by the property that they contain arbitrarily long blocks of consecutive integers, i.e., a set $T \subseteq \mathbb{N}$ is thick if and only if for every $h \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\{n, n+1, \ldots, n+h\} \subseteq T$.

Let us use $\mathscr{P}_{\text{syn}}$ to denote the family of all syndetic subsets of $\mathbb{N}$ and $\mathscr{P}_{\text{thick}}$ for the family of all thick subsets of $\mathbb{N}$.

Proposition 24. *The families $\mathscr{P}_{\text{syn}}$ and $\mathscr{P}_{\text{thick}}$ are dual, i.e., $\mathscr{P}_{\text{syn}}^* = \mathscr{P}_{\text{thick}}$ and $\mathscr{P}_{\text{thick}}^* = \mathscr{P}_{\text{syn}}$.*

Proof. Since any syndetic set has bounded gaps, it must have non-empty intersection with any thick set, because thick sets contain arbitrarily long intervals. From this, it follows that $\mathscr{P}_{\text{syn}} \subseteq \mathscr{P}_{\text{thick}}^*$. On the other hand, if a set intersects every thick set then its complement cannot be thick. If the complement is not thick then the set itself must have bounded gaps, i.e., it is syndetic. This implies $\mathscr{P}_{\text{thick}}^* \subseteq \mathscr{P}_{\text{syn}}$. In conclusion, we have $\mathscr{P}_{\text{syn}} = \mathscr{P}_{\text{thick}}^*$, which implies $\mathscr{P}_{\text{syn}}^* = \mathscr{P}_{\text{thick}}^{**} = \mathscr{P}_{\text{thick}}$ as desired. □

Definition 25. Sets belonging to $\mathscr{P}_{\text{syn}} \wedge \mathscr{P}_{\text{thick}}$ are called *piecewise syndetic* sets.

Piecewise syndetic sets are characterized by the property that they have bounded gaps on arbitrarily large intervals. Here is a more intuitive explanation of what this means. Let A be a subset of $\mathbb{N}$ and let a_n denote the n-th element of A, so that $a_1, a_2, a_3, \ldots$ becomes an increasing enumeration of elements in A. Then A is piecewise syndetic if and only if there exists some number $h \in \mathbb{N}$ with the following property: Somewhere in $A = \{a_1, a_2, a_3, \ldots\}$ there are two consecutive elements a_n, a_{n+1} whose distance $a_{n+1} - a_n$ is at most h. Somewhere else in A there are three consecutive elements a_m, a_{m+1}, a_{m+2} such that the distance between the first and the second $a_{m+1} - a_m$ and the distance between the second and the third $a_{m+2}-a_{m+1}$ are at most h. Then, somewhere else in the set, there exist four consecutive elements $a_k, a_{k+1}, a_{k+2}, a_{k+3}$ such that the distances $a_{k+1}-a_k$, $a_{k+2}-a_{k+1}$, $a_{k+3}-a_{k+2}$ are all at most h. And so on. This is another way of characterizing piecewise syndeticity.

Corollary 26. *Piecewise syndetic sets are partition regular.*

Proof. This follows by combining Proposition 21 and Proposition 24. □

Proposition 27. *Let $A \subseteq \mathbb{N}$ be piecewise syndetic. Then there exists a syndetic set L such that for any finite, non-empty $F \subseteq L$ the intersection*

$$\bigcap_{n\in F}(A-n) \tag{2.2.1}$$

is piecewise syndetic.

Proof. Since A is piecewise syndetic, there exist a syndetic set S and a thick set T so that $A = S\cap T$. Any thick set contains arbitrarily long intervals. Hence, by passing to a subset of T if necessary, we can assume without loss of generality that

$$T = [a_1, b_1]\cup[a_2, b_2]\cup[a_3, b_3]\cup\ldots$$

where $a_1 < b_1 < a_2 < b_2, \ldots \in \mathbb{N}$ with $b_n - a_n \to \infty$ as $n\to\infty$. Since S is syndetic, there exists $h\in\mathbb{N}$ for which $S\cup(S-1)\cup\ldots\cup(S-h+1)\supseteq\mathbb{N}$. Our goal is to construct a sequence $l_0 < l_1 < l_2 < \ldots \in \mathbb{N}$ such that $l_{n+1}-l_n \leqslant h$ for all $n\in\mathbb{N}$ and

$$\bigcap_{k=0}^{n}(A-l_k) \text{ is piecewise syndetic} \tag{2.2.2}$$

for all $n\in\mathbb{N}$. Once this task has been accomplished, we can take $L=\{l_n : n\in\mathbb{N}\}$ and we are done. Indeed L is syndetic because it has gaps bounded by h and (2.2.2) implies (2.2.1).

Let us now proceed with the construction of the sequence $l_0 < l_1 < l_2 < \ldots$, for which we use induction. Define $l_0 = 0$. If $l_0, l_1, \ldots, l_n$ have already been found, then l_{n+1} is constructed as follows: Define $A_n = \bigcap_{k=0}^{n}(A-l_k)$ and note that $A_n \subseteq A\subseteq T$. Since $S\cup(S-1)\cup\ldots\cup(S-h+1)\supseteq\mathbb{N}$, we also have $(S-l_n-1)\cup(S-l_n-2)\cup\ldots\cup(S-l_n-h)\supseteq\mathbb{N}$. In particular, by defining $A_{n,i} = A_n\cap(S-l_n-i)$ we get

$$A_n = A_{n,1}\cup\ldots\cup A_{n,h}.$$

Using Corollary 26, it follows from the fact that A_n is piecewise syndetic that for some $i\in\{1,\ldots,h\}$ the set $A_{n,i}$ is also piecewise syndetic. Define $l_{n+1} = l_n + i$ and note that

$$A_{n,i} = A_n\cap(S-l_{n+1}).$$

To finish the proof, let $T_{remainder} = T\setminus(T-l_{n+1})$ and note that

$$A_{n,i} = \big(A_{n,i}\cap(T-l_{n+1})\big)\cup\big(A_{n,i}\cap T_{remainder}\big),$$

because $A_{n,i} \subseteq T$. Since $A_{n,i}$ is piecewise syndetic, and the set

$$T_{remainder} \subseteq [b_1 - l_{n+1} + 1, b_1] \cup [b_2 - l_{n+1} + 1, b_2] \cup [b_3 - l_{n+1} + 1, b_3] \cup \ldots$$

is clearly not piecewise syndetic, we conclude from Corollary 26 that $A_{n,i} \cap (T - l_{n+1})$ must be piecewise syndetic. Thus the set

$$\begin{aligned}\bigcap_{k=0}^{n+1}(A - l_k) &= A_n \cap (A - l_{n+1}) \\ &= A_n \cap (S - l_{n+1}) \cap (T - l_{n+1}) \\ &= A_{n,i} \cap (T - l_{n+1})\end{aligned}$$

is piecewise syndetic, finishing the proof. □

From Proposition 27 we immediately obtain the following interesting corollary.

Corollary 28. *For any piecewise syndetic $A \subseteq \mathbb{N}$ there exist infinitely many $n \in \mathbb{N}$ such that $A \cap (A - n)$ is piecewise syndetic.*

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